`scatter`
The Reference Manual
version 1.3.10

Diab Jerius
Smithsonian Astrophysical Observatory
Chandra X-ray Center

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Contents

Chapter 1

Scattering Geometry

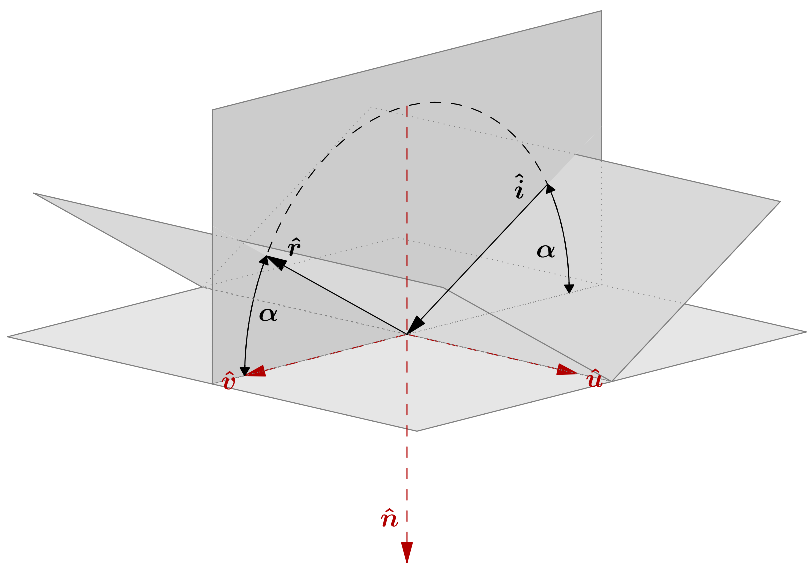

Figure 1.1: Incident and reflected rays and basis vectors for the reflecting surface

1.1 General Definitions

The incident ($\hat{\boldsymbol{i}}$) and reflected ($\hat{\boldsymbol{r}}$) rays are at the grazing angle α to the reflecting surface (plane), and the vector $\hat{\boldsymbol{n}}$ is normal to the reflecting plane and oriented such that

$$\begin{aligned} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{i}} &= \sin\alpha \\ \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} &= -\sin\alpha \end{aligned}$$

The reflecting plane is defined by the vectors

$$\begin{aligned} \hat{\boldsymbol{u}} &= \hat{\boldsymbol{r}} \times \hat{\boldsymbol{n}} / \cos\alpha \\ \hat{\boldsymbol{v}} &= \hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}} \end{aligned}$$

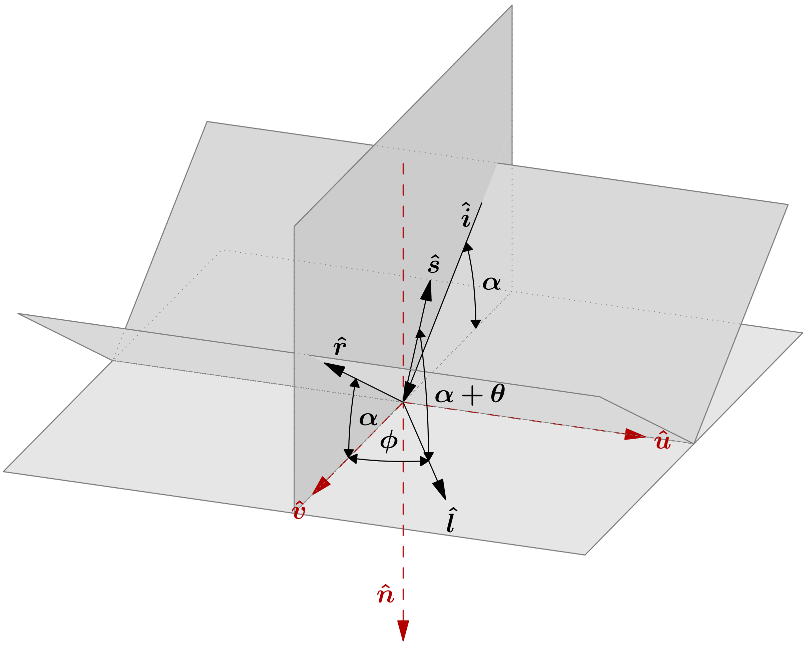

Figure 1.2: ϕ in $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$ plane

$\hat{\boldsymbol{v}}$ may also be written in terms of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{n}}$, to avoid computing an extra cross-product:

$$\begin{aligned}
\hat{\boldsymbol{r}} &= -\sin\alpha\ \hat{\boldsymbol{n}} + \cos\alpha\ \hat{\boldsymbol{v}} \\
\hat{\boldsymbol{v}} &= \frac{\hat{\boldsymbol{r}} + \sin\alpha\ \hat{\boldsymbol{n}}}{\cos\alpha}
\end{aligned}$$

The specular plane is that which contains both the incident and non-scattered reflected rays. The vectors $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{v}})$ form basis vectors for it.

The ray is scattered through angle θ relative to $\hat{\boldsymbol{r}}$, in the specular plane, and ϕ, relative to $\hat{\boldsymbol{r}}$. Which plane is ϕ in? Therein lies the rub.

The following relationships are useful to improve computational efficiency:

$$\begin{aligned}
\sin\alpha &= -\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} \\
\cos\alpha &= \sqrt{1 - \sin^2\alpha} \\
\cos(\alpha+\theta) &= \cos\alpha\ \cos\theta - \sin\alpha\ \sin\theta \\
\sin(\alpha+\theta) &= \sin\alpha\ \cos\theta + \cos\alpha\ \sin\theta
\end{aligned}$$

1.2 ϕ in $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$ plane

Let $\hat{\boldsymbol{l}}$ be a unit vector in the $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$ plane at an angle ϕ to the specular plane (see Fig. 1.2).

$$\hat{\boldsymbol{l}} = \cos\phi\ \hat{\boldsymbol{v}} + \sin\phi\ \hat{\boldsymbol{u}}$$

The scattered ray is then given by

$$
\begin{aligned}
\hat{\boldsymbol{s}} &= -\sin(\alpha+\theta)\ \hat{\boldsymbol{n}} + \cos(\alpha+\theta)\ \hat{\boldsymbol{l}} \\
&= -\sin(\alpha+\theta)\ \hat{\boldsymbol{n}} + \cos(\alpha+\theta)\left[\cos\phi\ \hat{\boldsymbol{v}} + \sin\phi\ \hat{\boldsymbol{u}}\right] \\
&= -\sin(\alpha+\theta)\ \hat{\boldsymbol{n}} + \cos(\alpha+\theta)\ \cos\phi\ \hat{\boldsymbol{v}} + \cos(\alpha+\theta)\ \sin\phi\ \hat{\boldsymbol{u}} \\
&= -\sin(\alpha+\theta)\ \hat{\boldsymbol{n}} + \cos(\alpha+\theta)\ \cos\phi\left[\frac{\hat{\boldsymbol{r}} + \sin\alpha\ \hat{\boldsymbol{n}}}{\cos\alpha}\right] + \cos(\alpha+\theta)\ \sin\phi\ \hat{\boldsymbol{u}} \\
&= -\sin(\alpha+\theta)\ \hat{\boldsymbol{n}} + \frac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}\ \hat{\boldsymbol{r}} + \frac{\cos(\alpha+\theta)\ \sin\alpha\ \cos\phi}{\cos\alpha}\ \hat{\boldsymbol{n}} + \cos(\alpha+\theta)\ \sin\phi\ \hat{\boldsymbol{u}} \\
&= \left(\frac{\cos(\alpha+\theta)\ \sin\alpha\ \cos\phi}{\cos\alpha} - \sin(\alpha+\theta)\right)\hat{\boldsymbol{n}} + \frac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}\ \hat{\boldsymbol{r}} + \cos(\alpha+\theta)\ \sin\phi\ \hat{\boldsymbol{u}}
\end{aligned}
$$

For computational efficiency, define

$$A = \frac{\cos(\alpha+\theta)}{\cos\alpha}$$

Then,

$$\hat{\boldsymbol{s}} = (A\ \sin\alpha\ \cos\phi - \sin(\alpha+\theta))\hat{\boldsymbol{n}} + A\,(\cos\phi\ \hat{\boldsymbol{r}} + \sin\phi\ \hat{\boldsymbol{r}} \times \hat{\boldsymbol{n}})$$

The $\hat{\boldsymbol{n}}$ component may be re-written for easier comparison with the other approaches:

$$
\begin{aligned}
&\frac{\cos(\alpha+\theta)\ \sin\alpha\ \cos\phi}{\cos\alpha} - \sin(\alpha+\theta) \\
&= \frac{\cos(\alpha+\theta)\ \sin\alpha\ \cos\phi - \sin(\alpha+\theta)\cos\alpha}{\cos\alpha} \\
&= \frac{(\cos(\alpha+\theta)\ \sin\alpha - \sin(\alpha+\theta)\ \cos\alpha)\cos\phi + \sin(\alpha+\theta)\ \cos\alpha\ \cos\phi - \sin(\alpha+\theta)\cos\alpha}{\cos\alpha} \\
&= \frac{-\sin\theta\ \cos\phi + \sin(\alpha+\theta)\ \cos\alpha(\cos\phi - 1)}{\cos\alpha} \\
&= -\frac{\sin\theta\ \cos\phi}{\cos\alpha} + \sin(\alpha+\theta)\,(\cos\phi - 1)
\end{aligned}
$$

1.3 ϕ in $(\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$ plane

$\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}$ is the scattered ray in the specular plane, i.e. $\phi = 0$ (see Fig. 1.3).

$$\hat{\boldsymbol{s}}_{\boldsymbol{\theta}} = \cos(\alpha+\theta)\ \hat{\boldsymbol{v}} - \sin(\alpha+\theta)\ \hat{\boldsymbol{n}}$$

$\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{u}}$ serve as basis vectors for a plane perpendicular to the specular plane. The final scattered ray is given by

$$
\begin{aligned}
\hat{\boldsymbol{s}} &= \cos\phi\ \hat{\boldsymbol{s}}_{\boldsymbol{\theta}} + \sin\phi\ \hat{\boldsymbol{u}} \\
&= \cos\phi\left[\cos(\alpha+\theta)\ \hat{\boldsymbol{v}} - \sin(\alpha+\theta)\ \hat{\boldsymbol{n}}\right] + \sin\phi\ \hat{\boldsymbol{u}} \\
&= \cos(\alpha+\theta)\ \cos\phi\ \hat{\boldsymbol{v}} - \sin(\alpha+\theta)\ \cos\phi\ \hat{\boldsymbol{n}} + \sin\phi\ \hat{\boldsymbol{u}} \\
&= \cos(\alpha+\theta)\ \cos\phi\left[\frac{\hat{\boldsymbol{r}} + \sin\alpha\ \hat{\boldsymbol{n}}}{\cos\alpha}\right] - \sin(\alpha+\theta)\ \cos\phi\ \hat{\boldsymbol{n}} + \sin\phi\ \hat{\boldsymbol{u}} \\
&= \frac{\sin\alpha\ \cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}\hat{\boldsymbol{n}} - \sin(\alpha+\theta)\ \cos\phi\ \hat{\boldsymbol{n}} + \frac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}\hat{\boldsymbol{r}} + \sin\phi\ \hat{\boldsymbol{u}}
\end{aligned}
$$

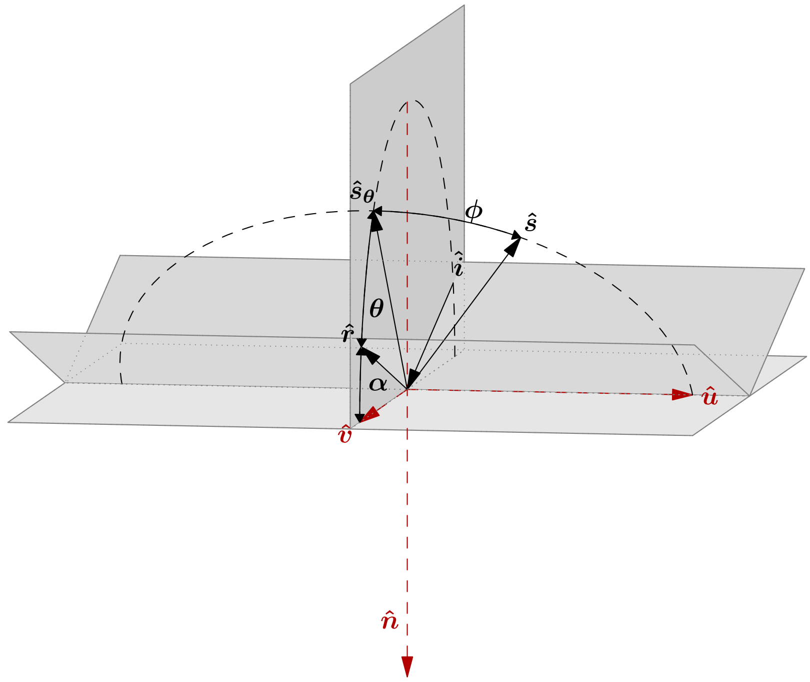

Figure 1.3: ϕ in $(\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$ plane

$$
\begin{aligned}
&= \frac{\cos\phi}{\cos\alpha}\left(\sin\alpha\,\cos(\alpha+\theta) - \cos\alpha\,\sin(\alpha+\theta)\right)\hat{\boldsymbol{n}} + \frac{\cos(\alpha+\theta)\,\cos\phi}{\cos\alpha}\hat{\boldsymbol{r}} + \sin\phi\,\hat{\boldsymbol{u}} \\
&= -\frac{\sin\theta\,\cos\phi}{\cos\alpha}\hat{\boldsymbol{n}} + \frac{\cos(\alpha+\theta)\,\cos\phi}{\cos\alpha}\hat{\boldsymbol{r}} + \sin\phi\,\hat{\boldsymbol{u}}
\end{aligned}
$$

1.4 ϕ in $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{u}})$ plane

$\hat{\boldsymbol{s}}_{\boldsymbol{\phi}}$ is the scattered ray in the plane perpendicular to the specular plane containing the reflected ray, i.e. $\theta = 0$ (see Fig. 1.4).

$$
\hat{\boldsymbol{s}}_{\boldsymbol{\phi}} = \cos\phi\,\hat{\boldsymbol{r}} + \sin\phi\,\hat{\boldsymbol{u}}
$$

Let $\hat{\boldsymbol{q}}$ be the vector perpendicular to $\hat{\boldsymbol{r}}$ and the $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{u}})$ plane.

$$
\begin{aligned}
\hat{\boldsymbol{q}} &= \hat{\boldsymbol{r}} \times \hat{\boldsymbol{u}} \\
&= -\sin\alpha\,\hat{\boldsymbol{v}} - \cos\alpha\,\hat{\boldsymbol{n}} \\
&= -\sin\alpha\left[\frac{\hat{\boldsymbol{r}} + \sin\alpha\,\hat{\boldsymbol{n}}}{\cos\alpha}\right] - \cos\alpha\,\hat{\boldsymbol{n}} \\
&= -\frac{\sin\alpha}{\cos\alpha}\,\hat{\boldsymbol{r}} - \left(\frac{\sin^2\alpha}{\cos\alpha} + \cos\alpha\right)\,\hat{\boldsymbol{n}} \\
&= -\frac{\sin\alpha}{\cos\alpha}\,\hat{\boldsymbol{r}} - \frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha}\,\hat{\boldsymbol{n}} \\
&= -\frac{\sin\alpha}{\cos\alpha}\,\hat{\boldsymbol{r}} - \frac{1}{\cos\alpha}\,\hat{\boldsymbol{n}}
\end{aligned}
$$

The final scattered ray is given by

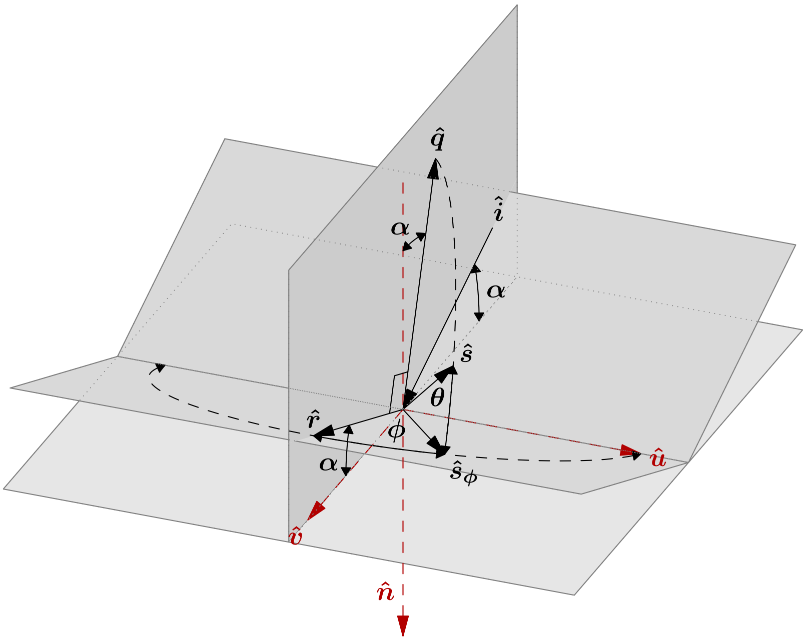

Figure 1.4: ϕ in $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{u}})$ plane

$$
\begin{aligned}
\hat{\boldsymbol{s}} &= \cos\theta\ \hat{\boldsymbol{s}}_{\boldsymbol{\phi}} + \sin\theta\ \hat{\boldsymbol{q}} \\
&= \cos\theta\ \cos\phi\ \hat{\boldsymbol{r}} + \cos\theta\ \sin\phi\ \hat{\boldsymbol{u}} + \sin\theta \left[-\frac{\sin\alpha}{\cos\alpha}\ \hat{\boldsymbol{r}} - \frac{1}{\cos\alpha}\ \hat{\boldsymbol{n}} \right] \\
&= -\frac{\sin\theta}{\cos\alpha}\ \hat{\boldsymbol{n}} + \cos\theta\ \cos\phi\ \hat{\boldsymbol{r}} - \frac{\sin\alpha\ \sin\theta}{\cos\alpha}\ \hat{\boldsymbol{r}} + \cos\theta\ \sin\phi\ \hat{\boldsymbol{u}} \\
&= -\frac{\sin\theta}{\cos\alpha}\ \hat{\boldsymbol{n}} + \left(\cos\theta\ \cos\phi - \frac{\sin\alpha\ \sin\theta}{\cos\alpha} \right)\ \hat{\boldsymbol{r}} + \cos\theta\ \sin\phi\ \hat{\boldsymbol{u}}
\end{aligned}
$$

The $\hat{\boldsymbol{r}}$ component may be re-written for easier comparison with the other approaches:

$$
\begin{aligned}
\cos\theta\ \cos\phi - \frac{\sin\alpha\ \sin\theta}{\cos\alpha} &= \frac{1}{\cos\alpha} \left(\cos\alpha\ \cos\theta\ \cos\phi - \sin\alpha\ \sin\theta \right) \\
&= \frac{1}{\cos\alpha} \left([\cos\alpha\ \cos\theta - \sin\alpha\ \sin\theta] \cos\phi + \sin\alpha\ \sin\theta\ \cos\phi - \sin\alpha\ \sin\theta \right) \\
&= \frac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha} + \frac{\sin\alpha\ \sin\theta (\cos\phi - 1)}{\cos\alpha}
\end{aligned}
$$

1.5 Summary

ϕ plane	$\hat{\boldsymbol{n}}$	$\hat{\boldsymbol{r}}$	$\hat{\boldsymbol{u}}$
$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$	$-\dfrac{\sin\theta}{\cos\alpha}\ \cos\phi + \sin(\alpha+\theta)\,(\cos\phi - 1)$	$\dfrac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}$	$\cos(\alpha+\theta)\ \sin\phi$
$(\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$	$-\dfrac{\sin\theta}{\cos\alpha}\ \cos\phi$	$\dfrac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha}$	$\sin\phi$
$(\hat{\boldsymbol{r}}, \hat{\boldsymbol{u}})$	$-\dfrac{\sin\theta}{\cos\alpha}$	$\dfrac{\cos(\alpha+\theta)\ \cos\phi}{\cos\alpha} + \dfrac{\sin\alpha\ \sin\theta(\cos\phi-1)}{\cos\alpha}$	$\cos\theta\ \sin\phi$

Or, in orthogonal coordinates,

ϕ plane	$\hat{\boldsymbol{n}}$	$\hat{\boldsymbol{u}}$	$\hat{\boldsymbol{v}}$
$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$	$-\sin(\alpha+\theta)$	$\cos(\alpha+\theta)\ \sin\phi$	$\cos(\alpha+\theta)\ \cos\phi$
$(\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$	$-\sin(\alpha+\theta)\ \cos\phi$	$\sin\phi$	$\cos(\alpha+\theta)\ \cos\phi$
$(\hat{\boldsymbol{r}}, \hat{\boldsymbol{u}})$	$-\sin(\alpha+\theta)\ \cos\phi + \cos\alpha\ \sin\theta(\cos\phi-1)$	$\cos\theta\ \sin\phi$	$\cos(\alpha+\theta)\ \cos\phi + \sin\alpha\ \sin\theta(\cos\phi-1)$

Chapter 2

Layout and Use of Scattering Tables

Scattering angles which would lead to the ray being scattered into the optic ($\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}} > 0$) are rejected.

Scattering angles which result in backscattered rays ($\mathrm{sgn}(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{v}}) \neq \mathrm{sgn}(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}})$) are rejected.

2.1 Old (LVS) tables

The scattering distribution is split into three regimes

- specular reflection (i.e. no scattering);
- a look-up table; and
- an analytical power law

The distribution is one-sided, so must be reflected to achieve both forwards and backwards scattering. The data are parameterized as a function of $E \sin\alpha$. They are stored as a FITS binary table with columns:

`ESA`	$E \sin\alpha$
`PDIST`	vector of scattering angles θ, in radians
`P0`	largest probability such that the reflection is still specular
`PTOP`	upper bound probability for the table lookup
`PDELTA`	spacing in probability of values in `PDIST`
`POWTOP`	power law exponent
`NORMTOP`	power law normalization factor

The `ESA` column must be in ascending order.
The interpolated scattering angle is given by

$$\theta(E \sin\alpha, P) = \theta_i + (\theta_{i+1} - \theta_i)\frac{E \sin\alpha - \mathtt{ESA}_i}{\mathtt{ESA}_{i+1} - \mathtt{ESA}_i}$$

where

$$\mathtt{ESA}_i < E \sin\alpha < \mathtt{ESA}_{i+1}$$

The scattering angle θ_m for a particular value of $E \sin\alpha$ ($\mathtt{ESA}_m$) is determined as follows:

$$\theta_m = \begin{cases} 0 & & P & \leq & \mathtt{P0}_m \\ \theta_{m,\mathrm{interp}} & \mathtt{P0}_m & < & P & \leq & \mathtt{PTOP}_m \\ (\mathtt{NORMTOP}_m(1-P))^{\mathtt{POWTOP}_m} & \mathtt{PTOP}_m & < & P & < & 0.99999 \\ (\mathtt{NORMTOP}_m(1-0.99999))^{\mathtt{POWTOP}_m} & 0.99999 & \leq & P \end{cases}$$

For each angle θ_n in $\texttt{PDIST}_m$,

$$P_n = n\,\texttt{PDELTA}_m + \texttt{P0}_m$$

and for a given input value of P,

$$\begin{array}{ccccc} P_j & < & P & < & P_{j+1} \\ j & < & j' & < & j+1 \end{array}$$

where $j' = (P - \texttt{P0}_m)/\texttt{PDELTA}_m$. This leads to

$$\begin{aligned} \theta_{m,\mathrm{interp}} &= \texttt{PDIST}_{m,j} + (\texttt{PDIST}_{m,j+1} - \texttt{PDIST}_{m,j})\frac{P - P_j}{P_{j+1} - P_j} \\ &= \texttt{PDIST}_{m,j} + (\texttt{PDIST}_{m,j+1} - \texttt{PDIST}_{m,j})(j' - j) \\ &= \texttt{PDIST}_{m,j}(1 - (j' - j)) + \texttt{PDIST}_{m,j+1}(j' - j) \end{aligned}$$

The out-of-plane scattering angle is determined via.

$$\phi = \sin\alpha\,\theta(E\,\sin\alpha, P')$$

2.2 New (PZ) tables

The scattering distribution is parameterized as a function of $E\,\sin\alpha$, and is composed of tables of scattering angles. Each table has the same number of angles, each of which have equal probability. The direction of the scattering is such that

$$\theta = \alpha_i - \alpha_s$$

where α_i and α_s are the incident and scattered grazing angles, respectively. θ is thus positive in the forward scattering direction.

The distribution is stored as a FITS binary table with columns

`ESA`	$E\sin\alpha$
`THETA`	vector of scattering angles θ, in seconds of arc

The `ESA` column must be in ascending order.

The interpolated scattering angle is given by

$$\theta(E\,\sin\alpha, P) = \theta_i + (\theta_{i+1} - \theta_i)\frac{E\,\sin\alpha - \texttt{ESA}_i}{\texttt{ESA}_{i+1} - \texttt{ESA}_i}$$

where

$$\texttt{ESA}_i < E\,\sin\alpha < \texttt{ESA}_{i+1}$$

and, given $P_n = n/N$ (where N is the number of elements in the table), and,

$$\begin{array}{ccccc} P_j & < & P & < & P_{j+1} \\ j & < & j' & < & j+1 \end{array}$$

where $j' = PN$. This leads to

$$\begin{aligned} \theta_m &= \texttt{THETA}_{m,j} + (\texttt{THETA}_{m,j+1} - \texttt{THETA}_{m,j})\frac{P - P_j}{P_{j+1} - P_j} \\ &= \texttt{THETA}_{m,j} + (\texttt{THETA}_{m,j+1} - \texttt{THETA}_{m,j})(j' - j) \\ &= \texttt{THETA}_{m,j}(1 - (j' - j)) + \texttt{THETA}_{m,j+1}(j' - j) \end{aligned}$$

The out-of-plane scattering angle is determined via.

$$\phi = \sin\alpha \times \theta(E\,\sin\alpha, P') \times \mathrm{sgn}(0.5 - P'')$$