Date: Feb. 5, 2004
To: L3 Distribution
From: F. Primini
Subject: Computing Flux and Flux Significance

1 Introduction

Here are simple aperture photometry formulae for computing photon flux and flux significance for a point source. For "flux significance" I will adopt the traditional "signal-to-noise ratio" definition, i.e., $Flux/\sigma_{Flux}$, where σ_{Flux} is the 1-sigma error in flux. Strictly speaking, it is only in the limit of Gaussian statistics that a single "1-sigma error" has meaning, but I'll defer discussion of this point to the end of the memo, and in what follows leave formulae for σ_{Flux} in general terms.

2 Definitions

$\mathbf{x_0,y_0}$	Position of Source (centroid)
S	Source Flux (ph cm^{-2} s^{-1})
n	Counts in Source Aperture
$\mathbf{A_S}$	Area in pixels of Source Aperture
$\mathbf{E_S}$	Average Exposure ($cm^2 s$) in Source Aperture
α	Fraction of PSF enclosed in Source Aperture, i.e., $\alpha = \int_{Source\ Aperture} psf(x_0, y_0, x, y)\, dxdy$
B	Background Density (counts $pixel^{-1}$)[1]
m	Counts in Background Aperture
$\mathbf{A_B}$	Area of Background Aperture
$\mathbf{E_B}$	Average Exposure in Background Aperture
β	Fraction of PSF enclosed in Background Aperture, i.e., $\beta = \int_{Background\ Aperture} psf(x_0, y_0, x, y)\, dxdy$

3 Simple Case: Isolated Source, $\beta = 0$

The situation is shown in Figure 1. I have drawn a detached background aperture for clarity; in reality, it may be attached to the source aperture. The assumption $\beta = 0$ means that no source counts are scattered into the background aperture, and could be realized by setting the background aperture radius to a large value (e.g., > r_{99}). Whether this would be practical or desirable is open to debate.

By inspection, the flux for the source is then given by

$$S = \left(\frac{1}{\alpha E_S}\right)\left(n - \frac{A_S}{A_B} m\right)$$

and the error on the flux is given by

$$\sigma_S^2 = \left(\frac{1}{\alpha E_S}\right)^2 \left(\sigma_n^2 + \left(\frac{A_S}{A_B}\right)^2 \sigma_m^2\right)$$

4 A More Complicated Case: Isolated Source, $\beta > 0$

In this case, the background aperture is sufficiently close to the source that some source counts are scattered into it. Now, S must be determined by solving two simultaneous linear equations:

$$n = \alpha E_S S + A_S B$$
$$m = \beta E_B S + A_B B$$

or

$$S = \frac{n - \frac{A_S}{A_B} m}{\alpha E_S - \beta E_B \frac{A_S}{A_B}}$$

Note this reduces to the equation for S in the previous section on setting $\beta = 0$. For simplicity, let

$$a = \frac{1}{\alpha E_S - \beta E_B \frac{A_S}{A_B}}$$

and

$$b = \frac{\frac{A_S}{A_B}}{\alpha E_S - \beta E_B \frac{A_S}{A_B}}$$

we can then write

$$S = an - bm$$

and

$$\sigma_S^2 = a^2 \sigma_n^2 + b^2 \sigma_m^2$$

5 General Case: Multiple, Overlapping Sources

Consider the case indicated in Figure 2. Here, each source aperture includes contributions not only from the source it encloses and background, but also possibly from nearby sources. For laziness sake, I've drawn all regions as simple, independent circles and annuli; in real life they may overlap, and will have to be adjusted by excluding parts to ensure that the counts n_1, n_2, m_1, and m_2 are statistically independent.

Generalizing the definitions for α and β in Section 2 to the case of multiple sources:

α_{ij}	Fraction of PSF for source i enclosed in source aperture for source j;
β_{ij}	Fraction of PSF for source i enclosed in background aperture for source j ;
E_{S_i}	Average Exposure ($cm^2 s$) in Source Aperture i;
A_{S_i}	Area in pixels of Source Aperture i ;
E_{B_j}	Average Exposure ($cm^2 s$) in Background Aperture j;
A_{B_j}	Area in pixels of Background Aperture j ;

We can then write, for the case of two sources as in figure 2:

$$
\begin{aligned}
n_1 &= \alpha_{11} E_{S_1} S_1 + \alpha_{21} E_{S_1} S_2 + A_{S_1} B \\
n_2 &= \alpha_{12} E_{S_2} S_1 + \alpha_{22} E_{S_2} S_2 + A_{S_2} B \\
m_1 &= \beta_{11} E_{B_1} S_1 + \beta_{21} E_{B_1} S_2 + A_{B_1} B \\
m_2 &= \beta_{12} E_{B_2} S_1 + \beta_{22} E_{B_2} S_2 + A_{B_2} B
\end{aligned}
$$

But the last two equations can be summed into one, namely,

$$\sum_j m_j = \sum_j \beta_{1j} E_{B_j} S_1 + \beta_{2j} E_{B_j} S_2 + \sum_j A_{B_j} B$$

where the sums are over the number of background regions. We now have 3 simultaneous linear equations, which we can solve for S_1, S_2, and B. The solutions will again be of the form

$$
\begin{aligned}
S_i &= a_i n_1 + b_i n_2 + c_i m \\
\sigma_{S_i}^2 &= a_i^2 \sigma_{n_1}^2 + b_i^2 \sigma_{n_2}^2 + c_i^2 \sigma_m^2
\end{aligned}
$$

The expression of a, b, and c in terms of the various α , β , etc. is left as an exercise for the reader. In general, for N sources, there will be N+1 simultaneous linear equations to solve, and S_i and $\sigma_{S_i}^2$ will be linear combinations of the counts n_i and $\sigma_{n_i}^2$, respectively. However, it probably doesn't pay to extend the analysis to N > ~ 2−3 since the assumption that B is constant will probably be violated.

6 Determining σ_n^2

All that remains is to specify how to calculate quantities of the form σ_n^2 , where n is small, so that Poisson statistics apply. As mentioned before, using a single number for "1 σ error" is only valid in the Gaussian limit, where σ^2 ~ n, but this is a poor approximation for small n. However, if we interpret s as the half-size of the two-sided 68.27% confidence limit about the true value, we can extend the concept more accurately for small n cases. The popular Gehrels approximation (Gehrels 1986, ApJ, 303, 336)

$$\sigma_n \approx 1 + \sqrt{n + 3/4}$$

is a better approximation than $\sqrt{n}$ (this is the approximation used in the Penn State Photometry code), but it overestimates the 68% confidence region. This is because the region is not symmetric about the true value. In fact, Gehrels provides two formulae, one for the lower bound, λ_l , and one for the upper bound, λ_u (see equations 7 and 11 in his paper; S=1 for the 68% confidence region):

$$\lambda_u = n + \sqrt{n+3/4} + 1$$
$$\lambda_l = n - \sqrt{n-1/4}$$

The following table illustrates the differences in the approximations for various values of n:

n	$\sigma_n = \sqrt{n}$	$\sigma_n = 1 + \sqrt{n+3/4}$	$\sigma_n = \frac{\lambda_u - \lambda_l}{2} = \frac{\sqrt{n+3/4} + \sqrt{n-1/4} + 1}{2}$
10	3.16	4.28	3.70
30	5.48	6.55	6.00
50	7.07	8.12	7.59
100	10.0	11.0	10.5

I suggest we use this last approximation for σ_n.

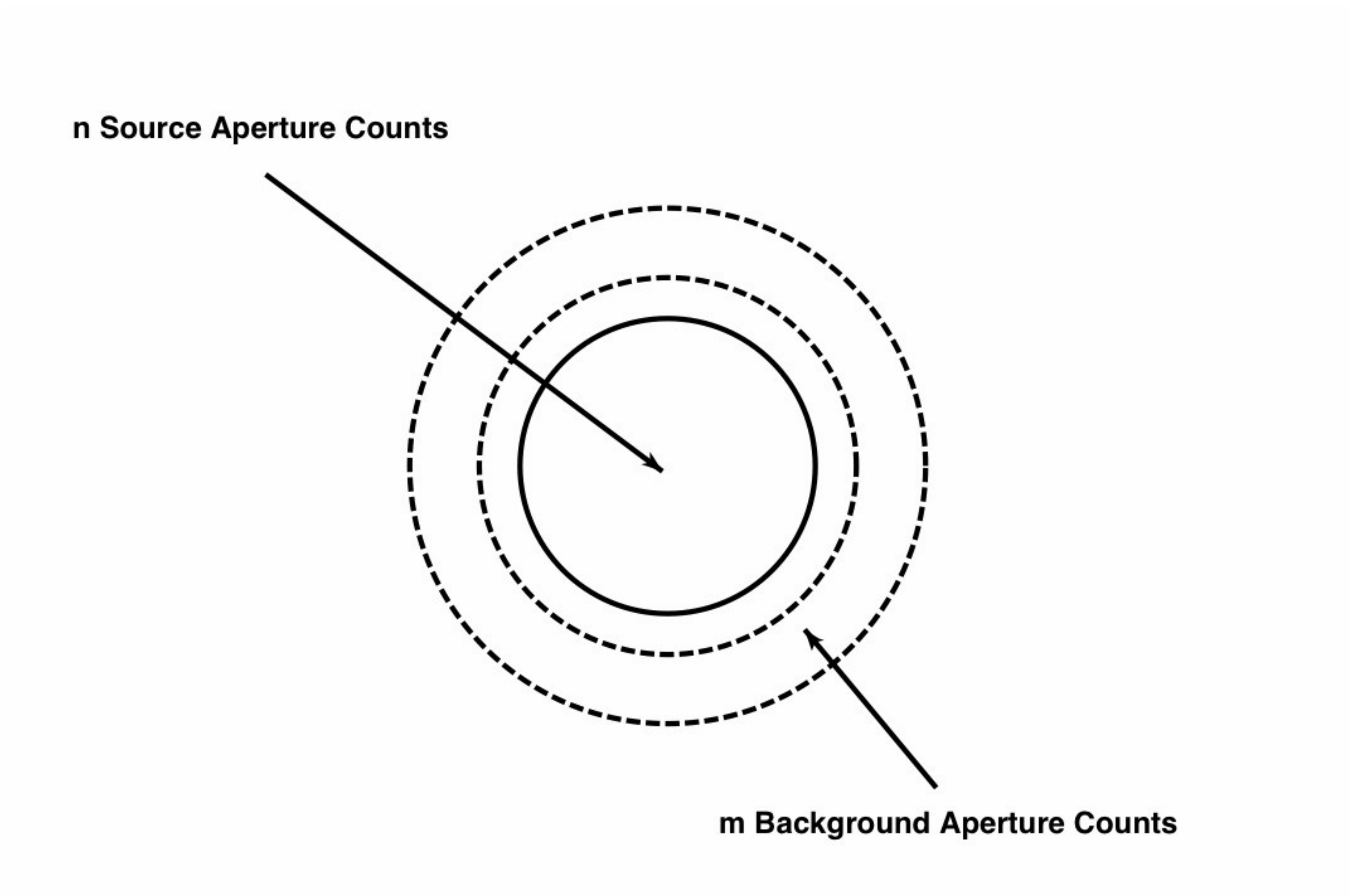

Figure 1.: A Single Isolated Point Source

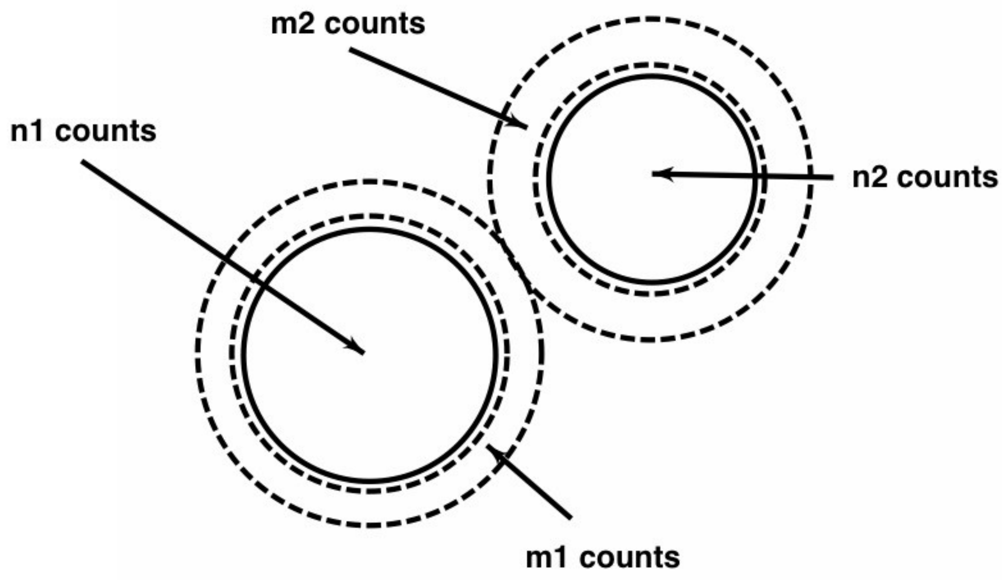

Figure 2.: Multiple Overlapping Sources

Footnotes:

[1]Since the background contains both cosmic and instrumental components, it shouldn't be expressed in the same units as source flux. I'm assuming here that the background is essentially flat (at least over the scale of the region of interest) and dominated by the non-cosmic component.